Quantum Algorithms for Binary Problems with Applications to Image Processing

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Doctor Thesis Defense

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April 23th | 10:00 am | MIC Arena

UNIVERSITÄT ZU LÜBECK
INSTITUTE OF MATHEMATICS
AND IMAGE COMPUTING

Section 1

Motivation & Outline

In this Thesis

Goal

Model as, and solve

$$\arg\min_{q\in\{0,1\}^n} \langle q|\mathbf{C}|q\rangle \,,$$

for

$$\mathbf{C} := \sum_{i=1}^{n} \mathcal{C}_{ii}\mathbf{Z}_i + \sum_{1\leq i<j\leq n} \mathcal{C}_{ij}\mathbf{Z}_i\mathbf{Z}_j,$$

given $\mathcal{C}_{ii}, \mathcal{C}_{ij} \in \mathbb{R}$, and $\mathbf{Z}_k$ being the Pauli-Z operator acting on qubit $k, k = 1, \ldots, n$.

Relates to: Ising's ground state-, QUBO-, Weighted maximum cut- problem.

Classical challenges:

- NP-Hard if non sub-modular
- Combinatorial, not differentiable

Quantum methods:

- Adiabatic quantum computing
- Universal quantum computing

Adiabatic- vs. Universal- QC

At any time $t \in [0, T]$, the evolution of the state vector $|\psi(t)\rangle$ of a quantum system obeys Schrödinger's equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \mathbf{H}(t) |\psi(t)\rangle ,$$

where $\mathbf{H}$ is a Hermitian operator known as the system-driven Hamiltonian.

Adiabatic- vs. Universal- QC

At any time $t \in [0, T]$, the evolution of the state vector $|\psi(t)\rangle$ of a quantum system obeys Schrödinger's equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \mathbf{H}(t) |\psi(t)\rangle ,$$

where $\mathbf{H}$ is a Hermitian operator known as the system-driven Hamiltonian.

Two computation paradigms

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \mathbf{H}(t) |\psi(t)\rangle$$

$$\mathbf{H}(t) = \mathcal{A}(t)\mathbf{H}(0) + \mathcal{B}(t)\mathbf{H}(T)$$

Adiabatic quantum computing
$\mathbf{H}(0)$: initial Hamiltonian
$\mathbf{H}(T)$: problem Hamiltonian

$$|\psi(t)\rangle = \mathbf{U}(t) |\psi(0)\rangle$$

Universal quantum computing
$\mathbf{U}(t)$: unitary operator
$\mathbf{U}(t)$: depends lonely on t

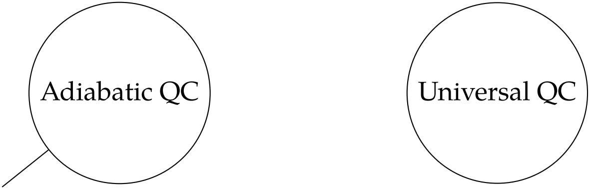
Contributions & Outline

1. Motivation & Outline

6. Bibliography

Contributions & Outline

1. Motivation & Outline



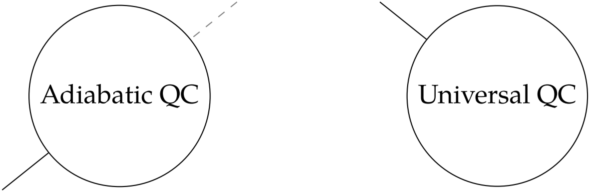
2.
Iterative Quantum
Transformation Estimation

6. Bibliography

Contributions & Outline

1. Motivation & Outline

3.
Variational Quantum Algorithm
for Ising Problems



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Contributions & Outline

1. Motivation & Outline

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Transformation Estimation

4.
Quantum Search
for Ising Problems

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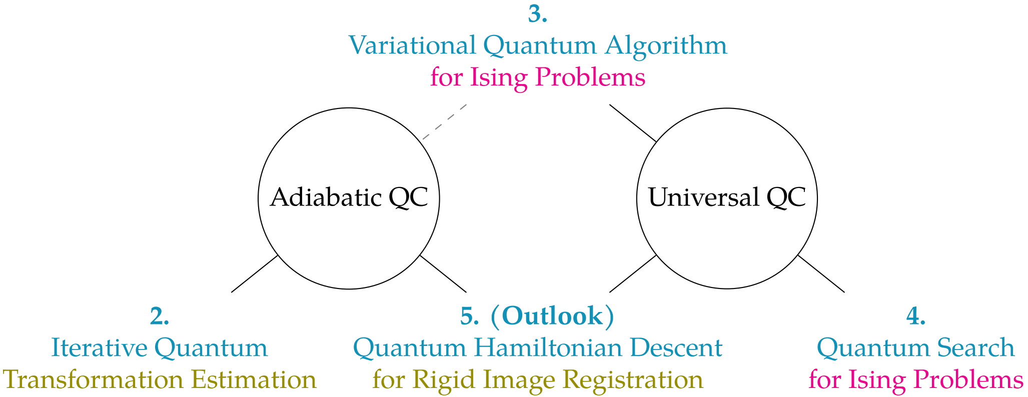
Contributions & Outline

1. Motivation & Outline



6. Bibliography

Section 2

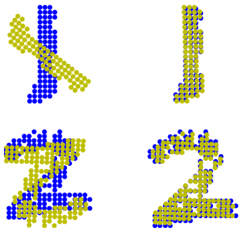
Quantum Transformation Estimation

Iterative Quantum Transformation Estimation

Goal

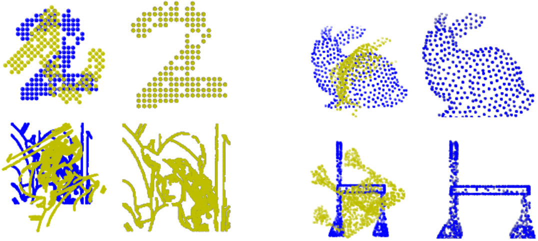
Given N pairs of points $(x_i, y_i) \in \mathbb{R}^d$, solve $\min_{\mathbf{R}\in SO(d)} \sum_{i=1}^{N} \|x_i - \mathbf{R}y_i\|^2$.

QA



(a) ***2D***.

IQT [New]



(a) *2D*. (b) *3D*.

QA: Golyanik and Theobalt. A Quantum Computational Approach to Correspondence Problems on Point Sets. CVPR, 2020.
IQT: Kuete Meli, Mannel, and Lellmann. An iterative quantum approach for transformation estimation from point sets. CVPR, 2022.

Iterative Quantum Transformation Estimation

Goal

Given N pairs of points $(x_i, y_i) \in \mathbb{R}^d$, solve $\min_{\mathbf{R}\in SO(d)} \sum_{i=1}^{N} \|x_i - \mathbf{R}y_i\|^2$.

	QA	**IQT [New]**
Optimization Variable	Matrix **R**	Parameter of **R**
Optimization Scheme	Fixed → fixed accuracy	Iterative → **flexible accuracy**
R orthogonal	×	✓
Number of qubits	**21 in 2D, 81 in 3D**	**10 in 2D, 15 in 3D**

QA: Golyanik and Theobalt. A Quantum Computational Approach to Correspondence Problems on Point Sets. CVPR, 2020.
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Iterative Quantum Transformation Estimation

Goal

Given N pairs of points $(x_i, y_i) \in \mathbb{R}^d$, solve $\min_{\mathbf{R}\in SO(d)} \sum_{i=1}^{N} \|x_i - \mathbf{R}y_i\|^2$.

IQT strategy: Write $\mathbf{R}$ as

$$\mathbf{R} = \exp(\mathbf{M}(v)),$$

where

in 2D for $v \in \mathbb{R}$	in 3D for $v := (v_1, v_2, v_3)^\top \in \mathbb{R}^3$
$\mathbf{M}(v) = v \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\mathbf{M}(v) = \|v\|_2 \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$

- Binarize using K-bit representation: $v_i = \sum_k 2^k q_{i_k}$
- Linearize using first order Taylor's expansion

Optimizing over $q \Rightarrow$ QUBO!!!

QA: Golyanik and Theobalt. A Quantum Computational Approach to Correspondence Problems on Point Sets. CVPR, 2020.
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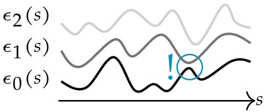
Adiabatic Quantum Theorem

D-Wave initial and problem Hamiltonians:

$$\mathbf{H}(0) = \mathbf{B}, \quad \mathbf{B} := \sum_{i=1}^{n} \mathbf{X}_i,$$

$$\mathbf{H}(T) = \mathbf{C}, \quad \mathbf{C} := \sum_{i=1}^{n} \mathcal{C}_{ii}\mathbf{Z}_i + \sum_{1\leq i<j\leq n} \mathcal{C}_{ij}\mathbf{Z}_i\mathbf{Z}_j.$$



Adiabatic theorem (roughly) [Albash et al. ‘2018]

Let $\mathbf{H}(t) := (1 - t/T)\mathbf{H}(0) + t\mathbf{H}(T)$ for $t \in [0, T]$ be an Hamiltonian with eigenstates $|\epsilon_j(t)\rangle$ to the eigenvalues $\epsilon_j(t)$, and so that $\epsilon_j(t) < \epsilon_{j+1}(t)$ for all $t \in [0, T]$ and $j \in \{0, 1, \cdots\}$. If the system is initialized in the state $|\epsilon_j(0)\rangle$, then Schrödinger's equation $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathbf{H}(t)|\psi(t)\rangle$ instantly keeps $|\psi(t)\rangle$ in $|\epsilon_j(t)\rangle$, provided that $\mathbf{H}(t)$ varies slowly enough.

Start in ground state $|+\rangle^{\otimes n}$ of $\mathbf{B}$ and end up in any ground state $|q^\star\rangle$ of $\mathbf{C}$!!!

IQT: Results

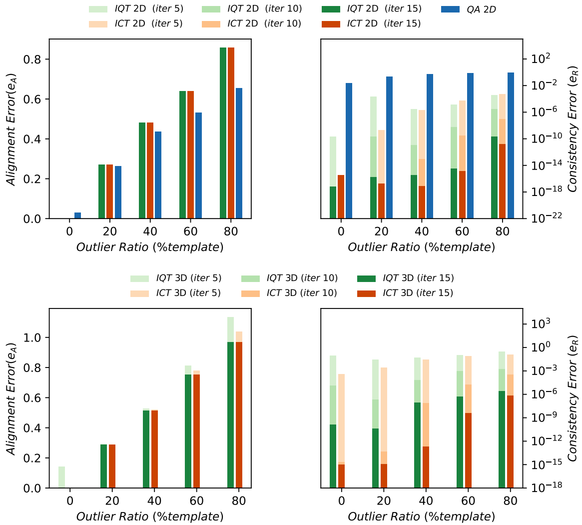
Metrics:

- Consistency error:

$$e_R := \|\mathbf{I} - \mathbf{R}^\top \mathbf{R}\|_F$$

- Alignment error:

$$e_A := \frac{\|\mathbf{X} - \mathbf{R}\mathbf{Y}\|_F}{\|\mathbf{X}\|_F}$$

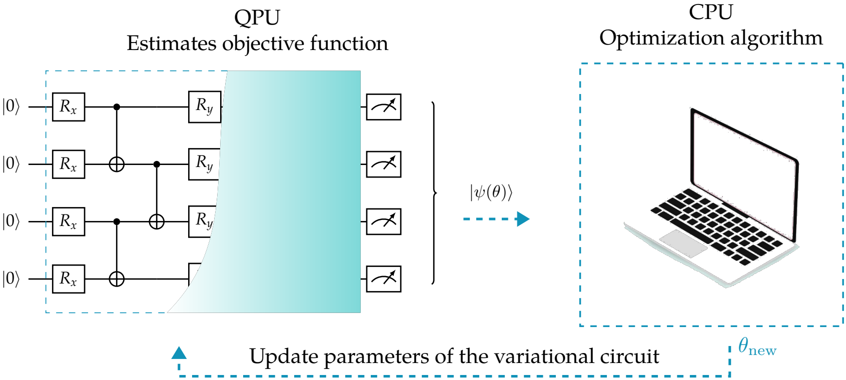


Kuete Meli, Mannel, and Lellmann. An iterative quantum approach for transformation estimation from point sets. CVPR, 2022.

Section 3

A Variational Quantum Algorithm for Ising Problems

Variational Quantum Computing (VQC)



VQC: *Cerez et al. Variational quantum algorithms. Nature, 2021.*
Peruzzo et al. A variational eigenvalue solver on a photonic quantum processor. Nature, 2014.
Wang et al. Variational quantum singular value decomposition. Quantum, 2021.

VQC for QUBO: A Concrete Case

Recall: we want to solve

$$\arg \min_{q \in \{0,1\}^n} \langle q|\mathbf{C}|q\rangle \,,$$

for

$$\mathbf{C} := \sum_{i=1}^{n} \mathcal{C}_{ii}\mathbf{Z}_i + \sum_{1\leq i<j\leq n} \mathcal{C}_{ij}\mathbf{Z}_i\mathbf{Z}_j .$$

Idea

Approximate $|q\rangle = \bigotimes_{i=1}^{n} |q_i\rangle \,, q_i \in \{0,1\}$ as $|\psi(\theta)\rangle = \sum_q \alpha_q(\theta)\,|q\rangle$ and solve

$$\arg \min_{\theta \in \Theta} \mathcal{L}(\theta), \quad \mathcal{L}(\theta) := \langle \psi(\theta)|\mathbf{C}|\psi(\theta)\rangle \,.$$

Idea (QAOA): *Farhi et al. A quantum approximate optimization algorithm. arXiv, 2014.*

VQC for QUBO: A Concrete Case

Recall: we want to solve

$$\arg\min_{q\in\{0,1\}^n} \langle q|\mathbf{C}|q\rangle,$$

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Idea

Approximate $|q\rangle = \bigotimes_{i=1}^{n} |q_i\rangle, q_i \in \{0,1\}$ as $|\psi(\theta)\rangle = \sum_q \alpha_q(\theta)|q\rangle$ and solve

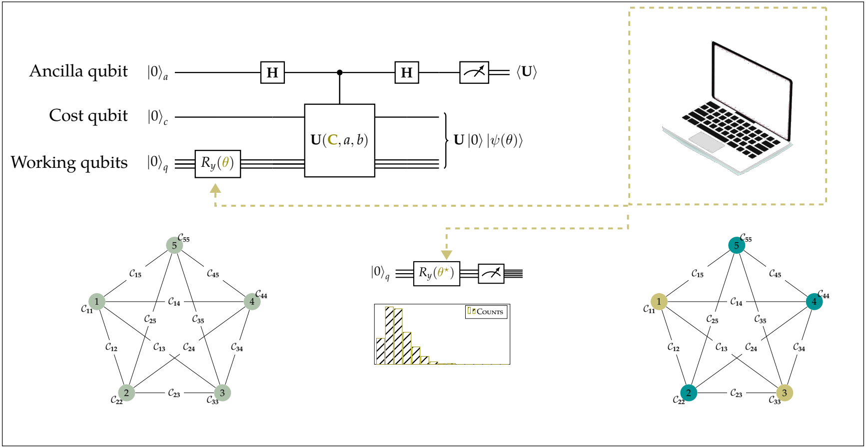
$$\arg\min_{\theta\in\Theta} \mathcal{L}(\theta), \quad \mathcal{L}(\theta) := \langle\psi(\theta)|\mathbf{C}|\psi(\theta)\rangle.$$

Block encoding [New]: Embed Hamiltonian **C** into a unitary operator **U** and solve

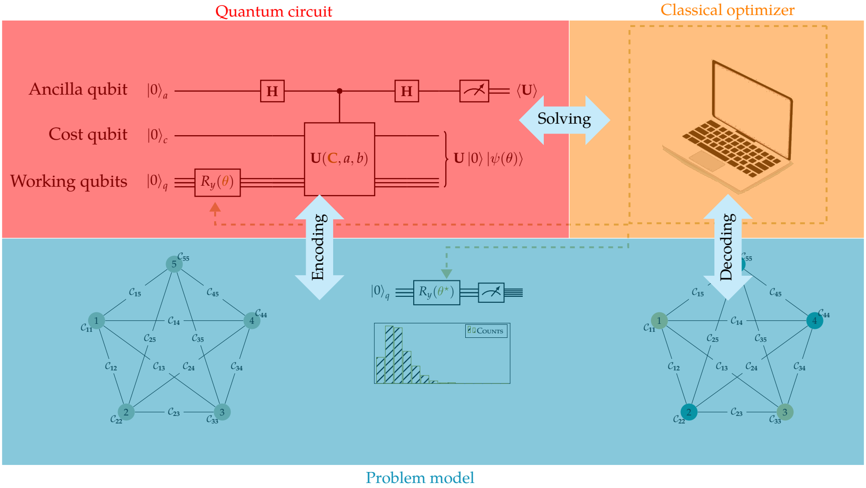
$$\arg\min_{\theta\in\Theta} \mathcal{L}(\theta), \quad \mathcal{L}(\theta) := \langle 0,\psi(\theta)|\mathbf{U}|0,\psi(\theta)\rangle.$$

Idea (QAOA): *Farhi et al. A quantum approximate optimization algorithm. arXiv, 2014.*
Block Encoding: *Kuete Meli et al. A universal quantum algorithm for weighted maximum cut and Ising problems. Springer, 2023.*

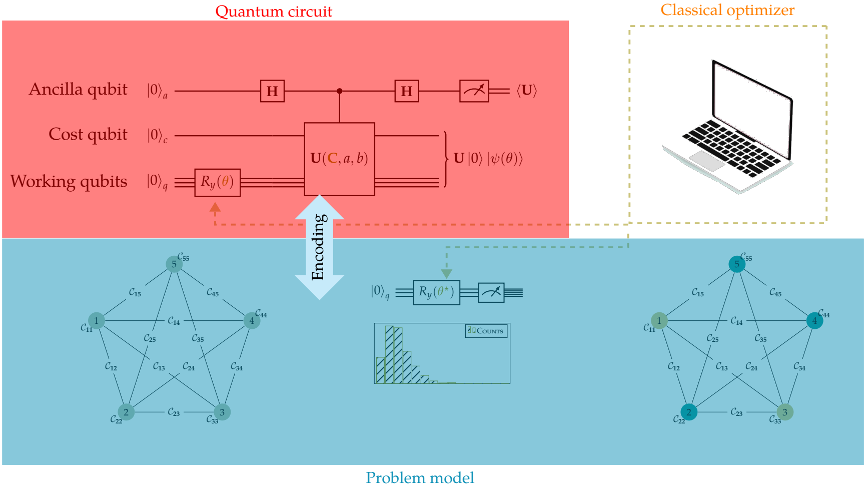
VQC for QUBO: Overview



VQC for QUBO: Overview



VQC for QUBO: Overview



VQC for QUBO: Block Encoding

Embed $\mathbf{C}$ into a $(2^{1+n}) \times (2^{1+n})$ unitary operator

$$\mathbf{U} := \mathbf{U}(\mathbf{C}, a, b) := \sum_q \mathbf{U}_{2\times 2}(q) \otimes |q\rangle\langle q|,$$

$$\mathbf{U}_{2\times 2}(q) := \begin{pmatrix} \cos(\langle q|\hat{\mathbf{C}}|q\rangle) & -\sin(\langle q|\hat{\mathbf{C}}|q\rangle) \\ \sin(\langle q|\hat{\mathbf{C}}|q\rangle) & \cos(\langle q|\hat{\mathbf{C}}|q\rangle) \end{pmatrix}, \quad \hat{\mathbf{C}} := a\mathbf{C} + b\mathbf{I}, \quad a, b \in \mathbb{R}.$$

- On a basis states it holds

$$\langle 0, q \,|\, \mathbf{U} \,|\, 0, q\rangle = \langle 0|\mathbf{U}_{2\times 2}(q)|0\rangle \otimes \langle q|q\rangle = \cos(\langle q|\hat{\mathbf{C}}|q\rangle).$$

- On an arbitrary state il holds

$$\langle 0, \psi(\theta) \,|\, \mathbf{U} \,|\, 0, \psi(\theta)\rangle = \sum_q \left|\alpha_q(\theta)\right|^2 \cos(\langle q|\hat{\mathbf{C}}|q\rangle).$$

Choose a, b so that $(a\mathbf{C} + b) \in [0, \pi]^{2^n}$ where $\cos$ ensures preserving order!

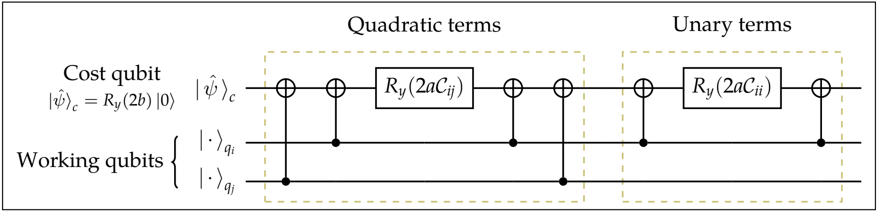
VQC for QUBO: Circuit Implementation

Using that

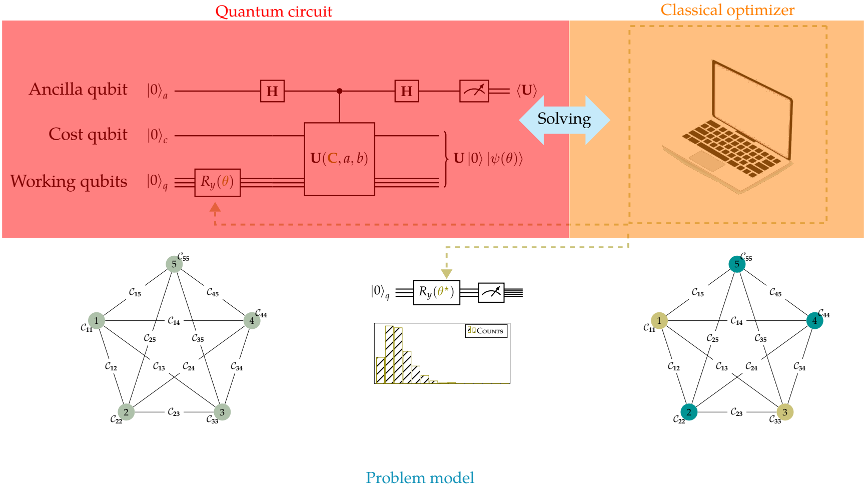
$$\langle q|\hat{\mathbf{C}}|q\rangle = a\,\langle q|\mathbf{C}|q\rangle + b, \quad \langle q|\mathbf{C}|q\rangle = \sum_{i=1}^{n}(-1)^{q_i}\mathcal{C}_{ii} + \sum_{1\le i<j\le n}(-1)^{q_i+q_j}\mathcal{C}_{ij},$$

we can implement $\mathbf{U}_{2\times 2}(q)$ as

$$\mathbf{U}_{2\times 2}(q) = R_y(\langle q|\hat{\mathbf{C}}|q\rangle) = \prod_{i=1}^{n} \mathbf{X}^{q_i}\cdot R_y(2a\mathcal{C}_{ii})\cdot \mathbf{X}^{q_i}\cdot \prod_{1\le i<j\le n} \mathbf{X}^{q_i+q_j}\cdot R_y(2a\mathcal{C}_{ij})\cdot \mathbf{X}^{q_i+q_j}\cdot R_y(2b).$$

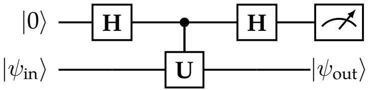


VQC for QUBO: Overview



VQC for QUBO: Hadamard Test

Consider the Hadamard test circuit and define operators $\mathbf{P}_\pm := \frac{1}{2}(\mathbf{I} \pm \mathbf{U})$.



Measurement with operators $\mathbf{P}_0 = |0\rangle\langle 0| \otimes \mathbf{I}$ and $\mathbf{P}_1 = |1\rangle\langle 1| \otimes \mathbf{I}$ yields

$$p(0) = \left\langle \psi_{\text{in}} \middle| \mathbf{P}_+^\dagger \mathbf{P}_+ \middle| \psi_{\text{in}} \right\rangle \qquad \text{and} \qquad p(1) = \left\langle \psi_{\text{in}} \middle| \mathbf{P}_-^\dagger \mathbf{P}_- \middle| \psi_{\text{in}} \right\rangle,$$

so that it holds $\operatorname{Re}\left(\langle \psi_{\text{in}} | \mathbf{U} | \psi_{\text{in}} \rangle\right) = p(0) - p(1)$.

In our application

$$\mathcal{L}(\theta) = \langle \psi_{\text{in}} | \mathbf{U} | \psi_{\text{in}} \rangle = \langle 0, \psi(\theta) | \mathbf{U} | 0, \psi(\theta) \rangle = \langle \psi(\theta) | \cos(\hat{\mathbf{C}}) | \psi(\theta) \rangle \in \mathbb{R}.$$

VQC for QUBO: Optimization

In an iterative process, we solve

$$\arg\min_{\theta \in \Theta} \mathcal{L}(\theta), \quad \mathcal{L}(\theta) := \langle 0, \psi(\theta) | \mathbf{U} | 0, \psi(\theta) \rangle ,$$

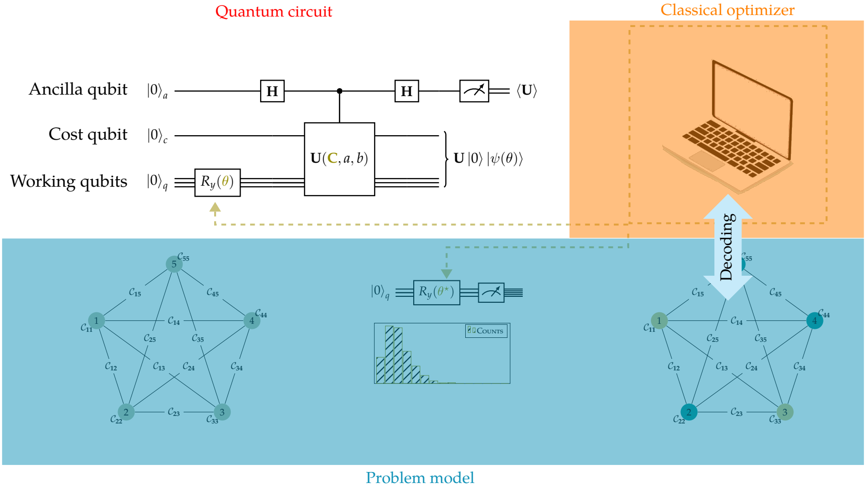
where we evaluate $\mathcal{L}(\theta)$ with Hadamard test.
Optimization with normalized gradient descent and decreasing step size:

$$\theta^{(k+1)} = \theta^{(k)} - \alpha_k \cdot \frac{\nabla_\theta \mathcal{L}(\theta^{(k)})}{\|\nabla_\theta \mathcal{L}(\theta^{(k)})\|_2^2}.$$

Parameter shift rule [Mitara et al. '2018]

$$\frac{\partial}{\partial \theta_i} \mathcal{L}(\theta) = \frac{1}{2} \left(\mathcal{L}\left(\theta + \frac{\pi}{2} e_i\right) - \mathcal{L}\left(\theta - \frac{\pi}{2} e_i\right) \right).$$

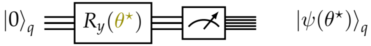
VQC for QUBO: Overview



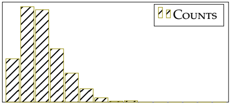
Decoding

Once optimal parameter vector $\theta^\star$ is found:

- Prepare and measure ansatz



- Measure and get count histogram



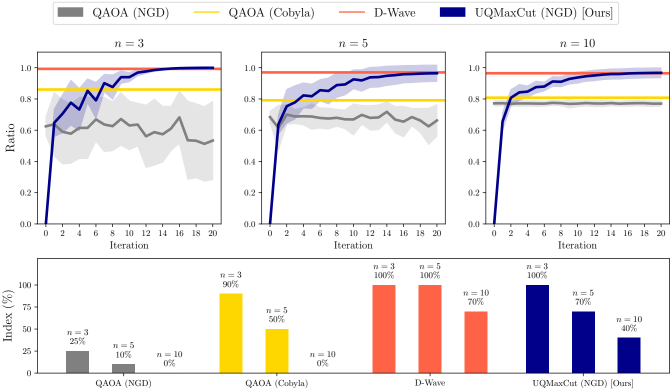
- Select solution (without loss of generality)

$$|\psi(\theta^\star)\rangle = \alpha_0 |0\rangle + \ldots + \alpha_{q^\star} |q^\star\rangle + \ldots + \alpha_{\max} |q_{\max}\rangle + \ldots + \alpha_{2^n-1} |2^n - 1\rangle$$

$$|\psi^\star\rangle = |q_{\max}\rangle$$

Results

Comparison against D-Wave:

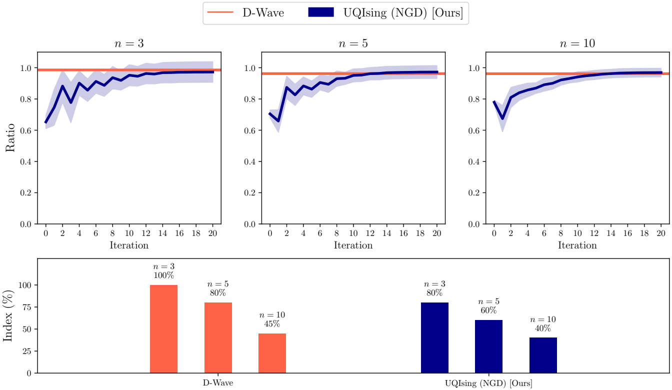


Metrics: $$\text{Ratio} = 1 - \frac{\langle \psi^\star | \mathbf{C} | \psi^\star \rangle - \mathcal{C}_{\min}}{\mathcal{C}_{\max} - \mathcal{C}_{\min}}, \quad \text{Index} = \# \{\psi^\star = q^\star\}.$$

Kuete Meli, Mannel, and Lellmann. A universal quantum algorithm for weighted maximum cut and Ising problems. Springer, 2023.

Results

Comparison against D-Wave:



Metrics: $\text{Ratio} = 1 - \frac{\langle \psi^\star | \mathbf{C} | \psi^\star \rangle - \mathcal{C}_{\min}}{\mathcal{C}_{\max} - \mathcal{C}_{\min}}$, $\text{Index} = \# \{ \psi^\star = q^\star \}$.

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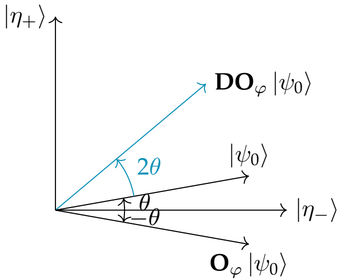
Section 4

Solving the Ising Problem by Quantum Search

Goal

Let $[N] := \{0, 1, \ldots, N-1\}$. Given $f : [N] \to \{0, 1\}$ find $q^\star \in [N]$ with $f(q^\star) = 1$.

Grover: Repeat $|\psi_{k+1}\rangle \leftarrow \mathbf{DO}_\varphi |\psi_k\rangle$ for $k = 0, \ldots, \lfloor \frac{\pi}{4\theta} \rfloor$ and chosen $|\psi_0\rangle$ $\Leftrightarrow$ rotate on 2D space formed by superposition state $|\eta_+\rangle$ of solutions and $|\eta_-\rangle$ of non-solutions:



Grover: Grover. A fast quantum mechanical algorithm for database search. Symposium on Theory of computing, 1996.

Grover's Quantum Search

Goal

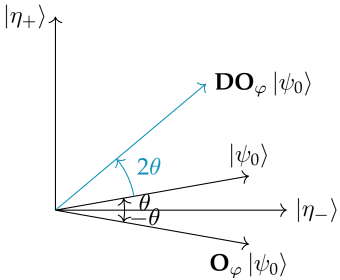
Let $[N] := \{0, 1, \ldots, N-1\}$. Given $f : [N] \to \{0,1\}$ find $q^\star \in [N]$ with $f(q^\star) = 1$.

Grover: Repeat $|\psi_{k+1}\rangle \leftarrow \mathbf{DO}_\varphi |\psi_k\rangle$ for $k = 0, \ldots, \lfloor \frac{\pi}{4\theta} \rfloor$ and chosen $|\psi_0\rangle \Leftrightarrow$ rotate on 2D space formed by superposition state $|\eta_+\rangle$ of solutions and $|\eta_-\rangle$ of non-solutions:

- $\mathbf{O}_\varphi$ is oracle operator defined for $\varphi(q) = \pi \cdot f(q)$ as

$$\mathbf{O}_\varphi |q\rangle := e^{i\varphi(q)} |q\rangle = \begin{cases} - & |q\rangle, \quad \text{if } f(q) = 1, \\ & |q\rangle, \quad \text{if } f(q) = 0, \end{cases}$$

so $\mathbf{O}_\varphi = (\mathbf{I} - 2 |\eta_+\rangle \langle \eta_+|)$



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Grover's Quantum Search

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Let $[N] := \{0, 1, \ldots, N-1\}$. Given $f : [N] \to \{0,1\}$ find $q^\star \in [N]$ with $f(q^\star) = 1$.

Grover: Repeat $|\psi_{k+1}\rangle \leftarrow \mathbf{D}\mathbf{O}_\varphi |\psi_k\rangle$ for $k = 0, \ldots, \lfloor \frac{\pi}{4\theta} \rfloor$ and chosen $|\psi_0\rangle \Leftrightarrow$ rotate on 2D space formed by superposition state $|\eta_+\rangle$ of solutions and $|\eta_-\rangle$ of non-solutions:

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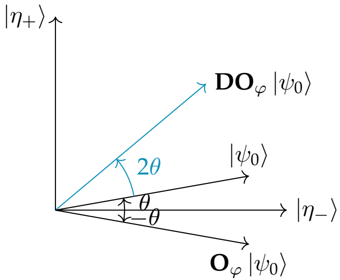
$$\mathbf{O}_\varphi |q\rangle := e^{i\varphi(q)} |q\rangle = \begin{cases} - \quad |q\rangle, & \text{if } f(q) = 1, \\ \quad |q\rangle, & \text{if } f(q) = 0, \end{cases}$$

so $\quad \mathbf{O}_\varphi = (\mathbf{I} - 2 |\eta_+\rangle \langle \eta_+|)$

- $\mathbf{D}$ is diffusion operator defined as

$$\mathbf{D} := -(\mathbf{I} - 2 |\psi_0\rangle \langle \psi_0|).$$



Grover: *Grover. A fast quantum mechanical algorithm for database search. Symposium on Theory of computing, 1996.*

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Goal

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Grover: Repeat $|\psi_{k+1}\rangle \leftarrow \mathbf{DO}_\varphi |\psi_k\rangle$ for $k = 0, \ldots, \lfloor \frac{\pi}{4\theta} \rfloor$ and chosen $|\psi_0\rangle \Leftrightarrow$ rotate on 2D space formed by superposition state $|\eta_+\rangle$ of solutions and $|\eta_-\rangle$ of non-solutions:

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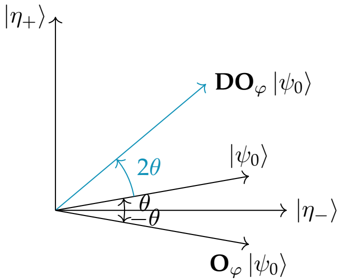
$$\mathbf{O}_\varphi |q\rangle := e^{i\varphi(q)} |q\rangle = \begin{cases} - \quad |q\rangle, & \text{if } f(q) = 1, \\ \quad |q\rangle, & \text{if } f(q) = 0, \end{cases}$$

so $\quad \mathbf{O}_\varphi = (\mathbf{I} - (1 - e^{i\pi}) |\eta_+\rangle \langle \eta_+|)$

- $\mathbf{D}$ is diffusion operator defined as

$$\mathbf{D} := -(\mathbf{I} - (1 - e^{i\pi}) |\psi_0\rangle \langle \psi_0|).$$



Grover: Grover. A fast quantum mechanical algorithm for database search. Symposium on Theory of computing, 1996.

Phase Matching

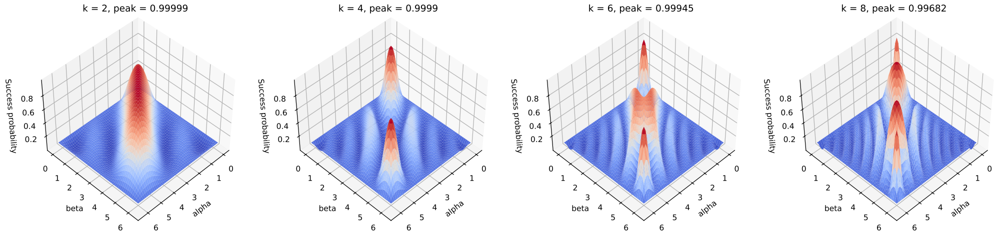
Will Grover's search still find a solution if we allow arbitrary phase rotations

$$\mathbf{O}_\varphi^\alpha := -\left(\mathbf{I} - \left(1 - e^{i\alpha}\right) |\eta_+\rangle \langle \eta_+| \right)$$

$$\text{and} \quad \mathbf{D}^\beta := -\left(\mathbf{I} - \left(1 - e^{i\beta}\right) |\psi_0\rangle \langle \psi_0|\right)?$$

Phase matching [Long et al. 1999].

Search possible in $\left\lfloor \frac{1}{\sin(\beta/2)} \left(\frac{\pi}{4\theta} - \frac{1}{2}\right)\right\rfloor$ iterations if $\alpha = \beta$ ($= \pi$ in Grover, optimal!).



NBAA: Non Boolean Amplitude Amplification

Goal

Given $f : [N] \to [0,1]$, resp. $\varphi : [N] \to [0,\pi]$, $\underset{q\in[N]}{\text{maximize}}\, f(q)$, resp., $\underset{q\in[N]}{\text{maximize}}\, \varphi(q)$.

NBAA: Repeat for $k = 0, \dots, \lfloor \frac{\pi}{2\theta} \rfloor$:

$$\begin{cases} |\psi_{k+1}\rangle & \leftarrow \mathbf{DO}_\varphi \, |\psi_k\rangle \,, \text{ if k odd,} \\ |\psi_{k+1}\rangle & \leftarrow \mathbf{DO}_\varphi^\dagger \, |\psi_k\rangle \,, \text{ if k even.} \end{cases}$$

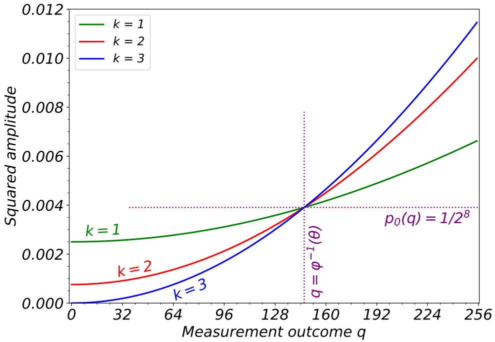
with $\mathbf{O}_\varphi$ and $\mathbf{D}$ as

$$\mathbf{O}_\varphi \, |0, q\rangle := e^{i\varphi(q)} \, |0, q\rangle \,,$$

$$\mathbf{O}_\varphi \, |1, q\rangle := e^{-i\varphi(q)} \, |1, q\rangle \,,$$

$$\mathbf{D} := -\left(\mathbf{I} - 2 \, |\psi_0\rangle \langle\psi_0|\right) .$$

$\Leftrightarrow \text{minimize}_{q\in[N]} \; \cos(\varphi(q))$!!!



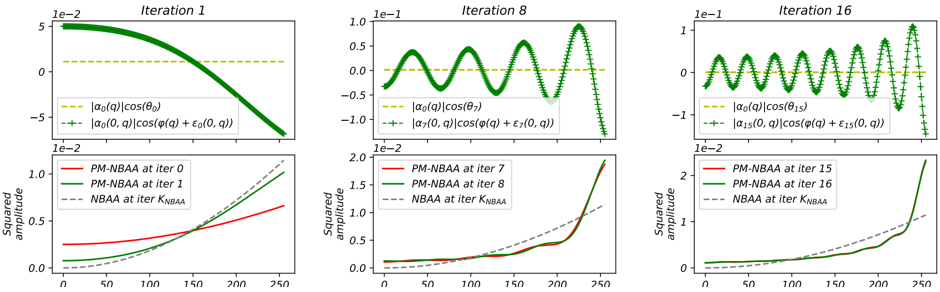
NBAA: *Shyamsundar. Non-boolean quantum amplitude amplification and quantum mean estimation. Springer, 2023.*

PM-NBAA: Phase-Matching NBAA

Satisfy two conditions [New]:

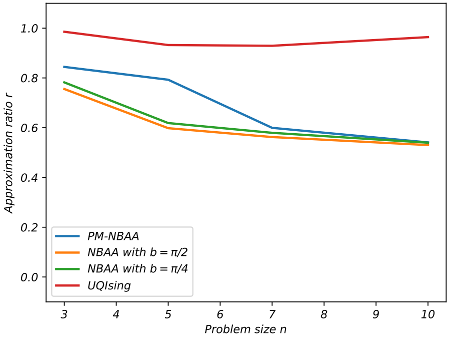
- Initial good overlap with the solution: $\langle q^\star|\psi_0\rangle > \langle q|\psi_0\rangle$ for all $q \neq q^\star$.
- Phase matching condition: $\varphi(q^\star) = \pm\pi$.

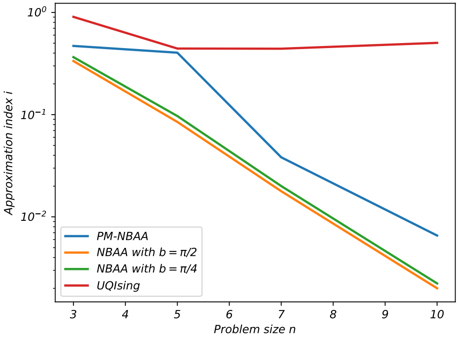
PM-NBAA: Repeat $|\psi_{k+1}\rangle \leftarrow \mathbf{DO}_{\varphi_{\text{PM}}} |\psi_k\rangle$, no distinctions between odd and even k:
$\Leftrightarrow$ Within K optimal iterations, $p_{k+1}(q^\star) \geq p_k(q^\star)$ and $p_{k+1}(q^\star) \geq p_{k+1}(q)$ for all $q \neq q^\star$!!!



Results on Ising's Problem

Comparison against UQIsing:





Metrics: $\text{Ratio} = 1 - \frac{\langle\psi^\star|\mathbf{C}|\psi^\star\rangle - \mathcal{C}_{\min}}{\mathcal{C}_{\max} - \mathcal{C}_{\min}}, \quad \text{Index} = \#\{\psi^\star = q^\star\}.$

Section 5

Quantum Hamiltonian Descent for Rigid Image Registration

A Quantum View on Optimization

Goal

Find

$$x^\star \in \arg\min_{x\in\mathcal{X}} f(x),$$

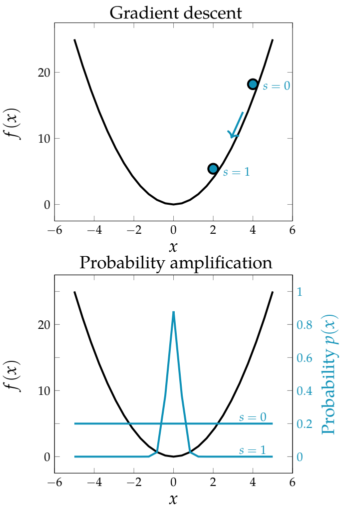
for some objective function $f : \mathcal{X} \to \mathbb{R}$.

1. Classical: Euler-Lagrange eq. with Lagrangian

$$\mathcal{L}(t, X_t, \dot{X}_t) := -e^{\chi_t} f(X_t) + e^{-\varphi_t}\left(\frac{1}{2}\|\dot{X}_t\|^2\right).$$

2. Quantum: Schrödinger eq. with Hamiltonian

$$\mathbf{H}(t) := e^{\chi_t} f + e^{\varphi_t}\left(-\frac{\hbar}{2}\Delta_x\right).$$



QHD: Quantum Hamiltonian Descent

QHD and convergence in the convex case [Leng et al. 2023]

Let f be a continuous differentiable convex function with a unique local minimizer $x^\star$ and the ideal scaling condition holds. Then, for any smooth initial wave function $|\psi(x,0)\rangle$, the solution $|\psi(x,t)\rangle$ at any time t of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(x,t)\rangle = \mathbf{H}(t)\,|\psi(x,t)\rangle$$

with the Hamiltonian

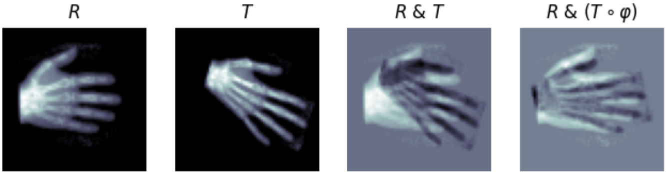
$$\mathbf{H}(t) := e^{\chi_t} f + e^{\varphi_t}\left(-\frac{\hbar}{2}\Delta_x\right)$$

such that $\lim_{t\to\infty} e^{\varphi_t/\chi_t} = 0$ satisfies

$$\int_{\mathcal{X}} f(x)\|\psi(x,t)\|^2 dx - f(x^\star) \leq O(e^{-\beta_t}).$$

Convergence also provable in the non-convex case under further assumptions.

Application on Rigid Image Registration



Goal

Given two images $R, T : [0, m] \times [0, n] \to \mathbb{R}_+$, solve

$$\arg\min_{\omega \in \Omega} \mathtt{SSD}(R, T \circ \varphi_\omega), \quad \mathtt{SSD}(R, T \circ \varphi_\omega) := \int_\Omega (R(x) - T(\varphi_\omega(x)))^2 \, dx,$$

with $\omega := (\omega_1, \omega_2, \omega_3)^\top, \Omega := [0, 2\pi] \times [-m, m] \times [-n, n]$ and $\varphi_\omega : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by

$$\varphi_\omega(x) := \begin{pmatrix} \cos(\omega_1) & -\sin(\omega_1) \\ \sin(\omega_1) & \cos(\omega_1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix}.$$

Trick [New]: Define state vector $|\omega(x, t)\rangle$ and let it evolve under QHD Hamiltonian.

Quantum Time Evolution

Evolution through solving the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\mathbf{U}(t) = \mathbf{H}(t)\mathbf{U}(t) \quad \text{and setting} \quad |\omega(x,T)\rangle = \mathbf{U}(T)\,|\omega(x,0)\rangle.$$

This generally can only be approximated through discretization

- in space as $|\omega(t)\rangle := |\omega(\cdot,t)\rangle = \sum_q \alpha_q(t)\,|\omega_q\rangle$,
- in time as $|\omega(j+1)\rangle := \exp\left(-i\frac{T}{r}\mathbf{H}(j\cdot\frac{T}{r})\right)|\omega(j)\rangle$ for $j = 0,\ldots,r$.

Update rule [Leng et al. 2023]

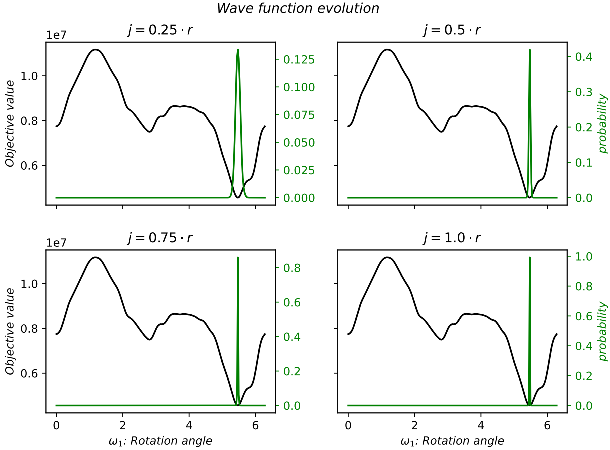
Defining a step size $s := \frac{T}{r}$ and letting $|\omega(0)\rangle$ being the uniform superposition state, we get the update rule

$$|\omega(j+1)\rangle = \Psi\cdot\exp\left(-is\mathcal{A}\left(js\right)\Sigma\right)\cdot\Psi^{\dagger}\cdot\exp\left(-is\mathcal{B}\left(js\right)f\right)|\omega(j)\rangle$$

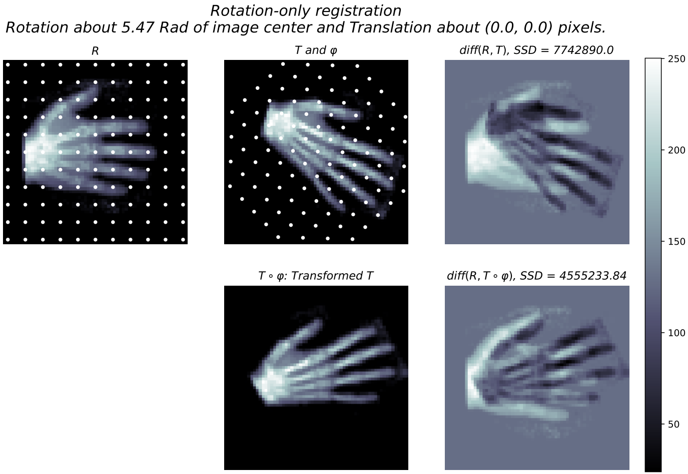
$j = 0,\ldots,r$. In our experiments, we set $T = 1$ and $r = 10^5$ update steps.

Results: Rotation Only

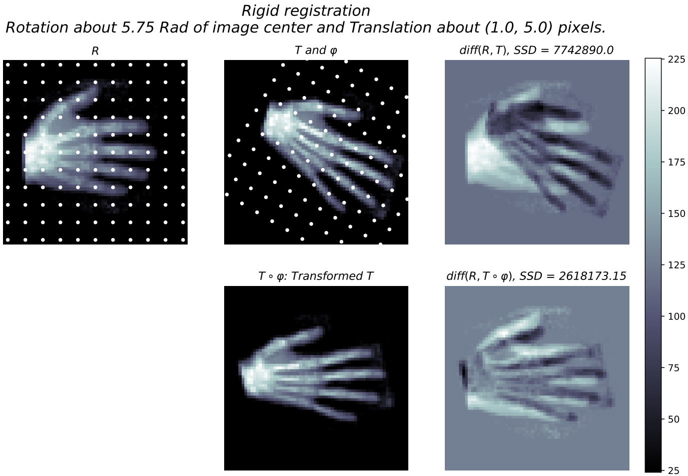
Wave function evolution under the action of QHD:



Results: Rotation Only



Results: Rotation and Translation



Section 6

Bibliography

Bibliography

Related author's publications:

- **Kuete Meli**, N., Mannel, F., and Lellmann, J. "An iterative quantum approach for transformation estimation from point sets". *CVPR*, 2022.
- **Kuete Meli**, N., Mannel, F., and Lellmann, J. "A universal quantum algorithm for weighted maximum cut and Ising problems". *Quantum Inf Process*, 2023.

Other relevant publications:

- Long et al. "Phase matching in quantum searching". *Physics Letters A*, 1999.
- Shyamsundar P. "Non-boolean quantum amplitude amplification and quantum mean estimation". *Quantum Inf Process*, 2023.
- Leng J. et al. Quantum hamiltonian descent. *Arxiv*, 2023.

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Thank You!