Quantum Hamiltonian Descent for Rigid Image Registration

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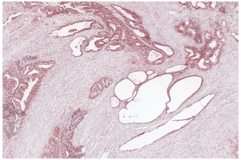
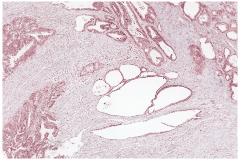
Rigid Image Registration

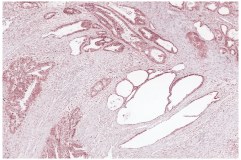
reference R

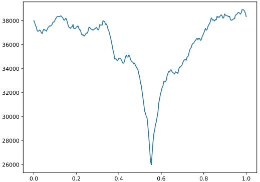


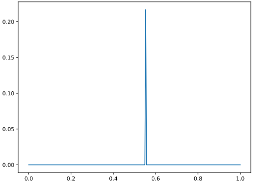
template T



QHD solution



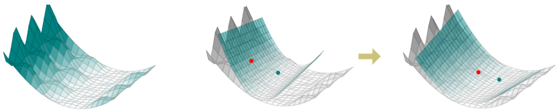




$$\min_{\phi\colon \mathbb{R}^d \to \mathbb{R}^d} \mathcal{D}(R, T \circ \phi) + \mathcal{R}(\phi), \quad \phi(x; t, \alpha)^{-1} = R_\alpha x + t, \quad \mathcal{R}(\phi) = 0.$$

Motivation

Optimizing non convex functions is challenging: Traditional methods only iteratively solve a **simple local** (quadratic) surrogate.



Kuete Meli et al., QuCOOP, CVPR 2025.

Quantum computers use superposition: They can “see” the **global** landscape of a function.

$$|\psi\rangle = \alpha_0 \quad |0\rangle + \cdots + \alpha_{2^n-1} \quad |2^n - 1\rangle$$
$$|\psi(f)\rangle = e^{i\theta f(0)} |0\rangle + \cdots + e^{i\theta f(2^n-1)} |2^n - 1\rangle$$

Can this global view ability of quantum computers be leveraged to escape local minima in optimization?

Background

QCs operate on quantum bits.

Qubit

A qubit is a one-particle system whose state vector is expressed as

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle ,$$

with

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} ,$$

$\alpha, \beta \in \mathbb{C}$ subject to $|\alpha|^2 + |\beta|^2 = 1$.

On multiple qubits, it holds:

- Kroneker product:

$$|\psi\rangle = \alpha_{000} |000\rangle + \alpha_{001} |001\rangle + \cdots + \alpha_{111} |111\rangle .$$

$|000\rangle$: $|0\rangle \otimes |0\rangle \otimes |0\rangle$ or $\binom{1}{0} \otimes \binom{1}{0} \otimes \binom{1}{0}$ or 0 in binary basis

$|001\rangle$: $|0\rangle \otimes |0\rangle \otimes |1\rangle$ or $\binom{1}{0} \otimes \binom{1}{0} \otimes \binom{0}{1}$ or 1 in binary basis

$|111\rangle$: $|1\rangle \otimes |1\rangle \otimes |1\rangle$ or $\binom{0}{1} \otimes \binom{0}{1} \otimes \binom{0}{1}$ or 7 in binary basis

- Entanglement:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\\0\\1\end{pmatrix} ;$$

no $|\psi_1\rangle , |\psi_2\rangle$ exist such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.

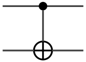
Quantum Operators and Evolution

The state vector $|\psi(t)\rangle$ of a quatum system at time t relates to the state $|\psi(0)\rangle$ via

$$|\psi(t)\rangle = \mathbf{U}(t)\,|\psi(0)\rangle\,,$$

where $\mathbf{U}$ is a unitary operator that uniquely depents on t.

Example of unitary operators:

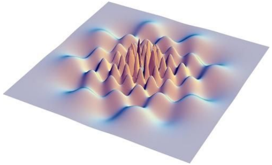
Gate name	Matrix form	Circuit	Example application
Hadamard	$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$		$\mathbf{H}\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
Phase gate	$\mathbf{P}(2\theta) = \begin{pmatrix}1 & 0\\ 0 & e^{-i\theta}\end{pmatrix}$		$\mathbf{P}\,|1\rangle = e^{-i\theta}\,|1\rangle$
Controlled-X on $|q_0 q_1\rangle$	$\mathbf{I}\otimes\mathbf{X}^{q_0} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$		$\mathbf{I}\otimes\mathbf{X}^{q_0}\,|00\rangle = |00\rangle$ $\mathbf{I}\otimes\mathbf{X}^{q_0}\,|10\rangle = |11\rangle$

Gate-based QC: *Nielsen and Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2010.*
Sutor. Dancing with Qubits: How quantum computing works and how it can change the world. Packt Publishing Ltd, 2019.

Continous Quantum Computing

In its continous form, a quantum state is a wave function $\psi(x, t)$ — a complex-valued function over $\mathbb{R}^d$, where $|\psi(x, t)|^2$ gives the **probability density** of finding a particle at position x and time t.

This wave function evolves according to the **Schrödinger equation**:



$$i\hbar \frac{\partial}{\partial t}\psi(x, t) = H(t)\psi(x, t),$$

where $H(t)$ is the **Hamiltonian** governing the system dynamics.

The goal is to design $H(t)$ to concentrate the wave function around the global minimum of a target function f — enabling optimization via quantum dynamics.

Continous QC: *Feynman et al. Quantum mechanics and path integrals, OUP Oxford 1966.*
Image source: `https://quantumphysicslady.org/wp-content/uploads/2019/03/Wavefunction-equationsource-wavefunction.png`

Quantum Hamiltonian Descent (QHD)

Quantum Hamiltonian Descent (QHD, Leng et al. 2023, 2025)

Consider the Schröndinger equantion

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H(t)\psi(x,t) \quad \text{under} \quad H(t) = e^{\chi_t} f(x) + e^{\phi_t}\left(-\frac{1}{2}\Delta\right)$$

with suitable real parameters ϕ_t, χ_t, Δ being the Laplacian operator. Under the condition that $e^{\phi_t}/e^{\chi_t} \overset{t\to\infty}{\to} 0$ and certain regularity assumptions on f and the Hamiltonian, it holds

$$\lim_{t\to\infty}\int_X |\psi(x,t)|^2 f(x) dx = \min f.$$

Measuring the state after a sufficiently long time t will yields a near-optimal solution x with high probability.

Continuous-Time Dynamics of Gradient Methods

Most gradient methods are discretizations of continuous-time flows:

Method	Update rule	Continuous limit ($\alpha \to 0$)
Gradient Descent	$x_{k+1} = x_k - \alpha \nabla f(x_k)$	$\dot{X}_t = -\nabla f(X_t)$
Nesterov	$x_{k+1} = x_k + \mu(x_k - x_{k-1}) - \alpha \nabla f(\cdots)$	$\ddot{X}_t + \frac{3}{t}\dot{X}_t = -\nabla f(X_t)$
Newton	$x_{k+1} = x_k - \alpha(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$	$\dot{X}_t = -(\nabla^2 f(X_t))^{-1}\nabla f(X_t)$

These flows X minimize an **action functional** (Wibisono et al., 2016):

$$S(X) = \int_t \mathcal{L}(t, X_t, \dot{X}_t)\, dt, \quad \mathcal{L} = e^{\alpha_t + \gamma_t}\left(\beta(t) f(X_t) + \mathrm{Dist}(X_t + \alpha(t)\dot{X}_t, X_t)\right),$$

by optimality condition $\nabla_X S(X) = 0$ yielding Euler–Lagrange Eq. $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) - \frac{\partial \mathcal{L}}{\partial X} = 0$.

Compare with QHD:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = H(t)\psi(x,t), \quad H(t) = e^{\chi_t} f(x) + e^{\phi_t}\left(-\tfrac{1}{2}\Delta\right)$$

The QHD Algorithm

Considering

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H(t)\psi(x,t), \quad H(t) = e^{\chi_t} f(x) + e^{\phi_t}\left(-\frac{1}{2}\Delta\right),$$

a spatial, then temporal discretization yield

$$i\hbar\frac{\partial}{\partial t}\psi_t = \left(e^{\chi_t}\hat{F} + e^{\phi_t}\hat{L}\right)\phi_t, \quad \text{then} \quad \psi_{j+1} = \exp(-ia_j \mathrm{dt}\hat{L} - ib_j \mathrm{dt}\hat{F})\,\psi_j,$$

where $a_j = e^{\phi_{j\cdot \mathrm{dt}}}$ and $b_j = e^{\chi_{j\cdot \mathrm{dt}}}$. Further simplifications apply Trotter-Suzuki as

$$\exp(-ia_j \mathrm{dt}\hat{L} - ib_j \mathrm{dt}\hat{F}) \approx \exp(-ia_j \mathrm{dt}\hat{L})\exp(-ib_j \mathrm{dt}\hat{F}),$$

and diagonalization $\hat{L} = \mathcal{F}D\mathcal{F}^{-1}$ for efficient exponentiation, $\mathcal{F}$ being the QFT.

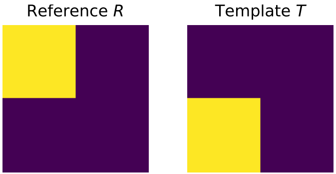
QHD Iteration

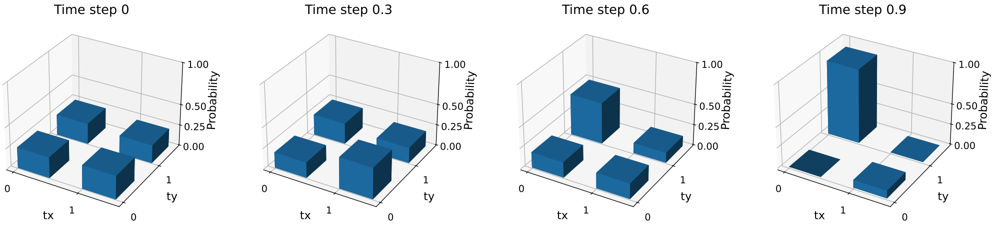
$$|\psi_{j+1}\rangle \coloneqq \mathcal{F}\exp(-ia_j \mathrm{dt}\hat{D})\mathcal{F}^{-1}\exp(-ib_j \mathrm{dt}\hat{F})\,|\psi_j\rangle\,, \quad |\psi_0\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle\,.$$

Example Implementation

Five registers are initialized as $|\psi_0\rangle = \sum_{t_x,t_y} |0\rangle_R |0\rangle_T |0\rangle_{\text{SSD}} |t_x\rangle_{t_x} |t_y\rangle_{t_y}$.

- $|R\rangle_R = |1000\rangle\ ,\ \ |T\rangle_T = |0010\rangle\ .$
- $|T \circ \phi(t_x, t_y)\rangle$: controlled shifts on $|T\rangle$.
- SSD phases $e^{-ib_0 \cdot \text{dt} \cdot \text{SSD}(t_x,t_y)}$ using:
  - $|a\rangle |b\rangle |\text{SSD}\rangle_{\text{SSD}} \mapsto |a\rangle |b\rangle |\text{SSD} + (a^2 + b^2 - 2ab)\rangle_{\text{SSD}}$,
  - $\bigotimes_{i=0}^{K} P(-b_0 \cdot \text{dt} \cdot 2^i) |q_0 q_1 \cdots q_K\rangle_{\text{SSD}}$.



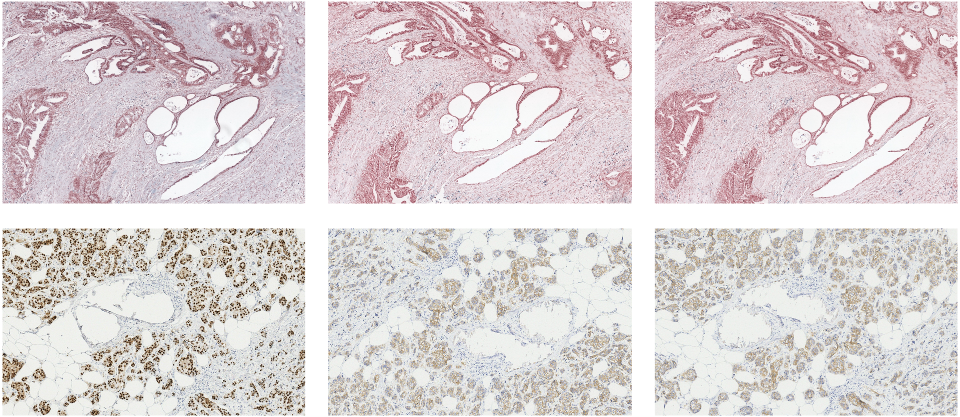


Title

reference R	template T	QHD solution	global optimium

Title

reference R | template T | QHD solution



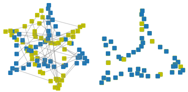
Title

Dataset	$p(x^*)$	$p(\bar{x})$	$\|x^* - \bar{x}\|_{\ell^1}$
balloon3	52.55%	56.68%	3
playground	4.18%	7.60%	4
dynamicFace	53.31%	57.45%	3
skating	25.38%	41.47%	5
coard17b	79.62%	79.62%	0
breast5a	58.93%	58.93%	0
breast5b	41.61%	41.79%	1
lung-lesion2	78.57%	78.57%	0

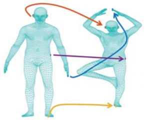
Further Interests

Adiabatic quantum computers can solve QUBO problems:

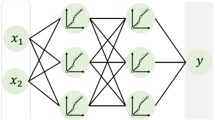
$$\min_{x \in \{0,1\}^n} \quad x^\top Q x$$



Correspondence problems
Kuete Meli et al., CVPR 2025



Matching problems
Benkner et al., 3DV 2020



Quantized neural nets
Li et al., ArXiv 2025

Quantum computers “see” the global landscape of a function:
Useful in optimization?

$$|\psi\rangle = \alpha_0 \quad |0\rangle + \cdots + \alpha_{2^n-1} \quad |2^n - 1\rangle$$
$$|\psi(f)\rangle = f(0)\,|0\rangle + \cdots + f(2^n - 1)|2^n - 1\rangle$$

Can we enhance computer vision solutions by leveraging quantum methods?
Enhancement can be in term of complexity, time-to-solution, memory demand, ...

Thanks!
Questions?